A New 3D, Fully Parallel, Unstructured AMR MHD High Order Godunov Code for Modeling Sun-Earth Connection Phenomena

Daniel S. Spicer\textsuperscript{a}, Hong Luo\textsuperscript{b}, John E Dorband\textsuperscript{c}, Kevin M. Olson\textsuperscript{a}, Peter J MacNeice\textsuperscript{d}

\textsuperscript{a}Department of Physics, Drexel University, Philadelphia, PA 20771, USA
\textsuperscript{b}Department of Mechanical and Aerospace Engineering, North Carolina State University, Campus Box 7910, Raleigh, NC 27695-7910, USA
\textsuperscript{c}Department of Computer Science and Electrical Engineering, University of Maryland, Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250 USA
\textsuperscript{d}Code 6740, NASA/GSFC Greenbelt, MD 20873 USA

Abstract

This paper describes a new node centered upwind finite element code, \textit{Adapt3DMHD}, that solves the compressible magnetohydrodynamics (MHD) equations on an unstructured mesh. In this scheme, the spatial discretization is accomplished by an edge based finite element formulation using Roe’s flux difference splitting. A MUSCL approach is used to achieve high order accuracy. Solutions are advanced in time by a multi-stage Runge-Kutta time stepping scheme. The algorithm has been tested and validated on many CFD test problems and real physical problems. Due to the solenoid constraint on the magnetic field $\mathbf{B}$, $\nabla \cdot \mathbf{B} = 0$, the requirement that the Lorentz force parallel to the magnetic field must vanish, $\mathbf{B} \cdot \nabla \mathbf{F} = 0$, and the likelihood there are regions in the flow domain where the magnetic energy density, $\mathbf{B}^2/2$, is often many orders of magnitude greater than the internal energy density it is not possible to directly apply conservative schemes to the MHD equations without causing violations of the solenoidal constraint, violations of momentum conservation, and obtaining negative pressures. We briefly discuss the measures we use to insure $\nabla \cdot \mathbf{B} = 0$, $\mathbf{B} \cdot \nabla \mathbf{F} = 0$, and non-negative pressures. Results from various test problems are presented as well as those from solar and space physics.

Keywords:
1. Introduction

The principal objective of this paper is to introduce to the Solar, Heliosphere, and Space Physics communities a new fully parallel adaptive 3D magnetohydrodynamics (MHD) code that utilizes an unstructured mesh made up of node based control volumes. While the majority of MHD codes reported in the Solar, Heliosphere, and Space Physics literature are structured, that is, they use meshes that are simply connected by incrementing the local cell index (i,j,k), our new code, Adapt3DMHD requires a lookup table which allows the use of highly complex internal and external flow boundaries, a capability difficult to implement with structured codes unless cut-cells are utilized, which to our knowledge have never been successfully used in MHD codes. Modeling a realistic solar atmosphere requires a 3D spherical or ellipsoidal surface on which solar active regions can be allowed to evolve and move due to realistic solar differential rotation. Global modeling of a realistic solar atmosphere also requires a numerical approach that will permit improved physical models be easily integrated into the overall code, vastly increasing the model functionality and for this reason alone justifies the added complexity of an unstructured numerical model. This increased functionality includes but is not limited to: a), the study of flows around complex boundaries representing complex objects embedded in the domain of the flow; b), unstructured meshes allow multiple connected domains, permitting the use of different physical approximations within each connected domain, but with shared boundary points and shared boundaries that can deform or “breathe” in response to internal and external forces, which we have dubbed ”virtual boundaries”; and c), the ability to permit the grid points making up a surface boundary to act as an independent grid on which two dimensional physical effects can be modeled. Examples of a) are the Jupiter-Io problem, the coupling of the solar interior to the solar atmosphere, a star orbiting a star, and the Sun-Earth system. An example of b) is a set of concentric spherical shells, each shell representing a different physical domain. Since each boundary shared between each shell also shares common boundary points, various physical domains representing the thermosphere, the ionosphere, the plasmasphere, and the magnetosphere, can be consistently modeled and coupled without the overhead of coupling independently designed models. An example of c), is the magnetosphere-ionosphere coupling, which has an MHD domain with a simple ionospheric model on the spherical surface representing the inner boundary of the MHD domain.
2. Governing Equations

The MHD equations governing unsteady compressible inviscid flows can be expressed in conservative form as

$$\frac{\partial Q_i(x, t)}{\partial t} + \frac{\partial F_{ij}(Q(x, t))}{\partial x_j} = 0,$$

in $\Omega$ (1)

where, $\Omega$ is a bounded connected domain in $\mathbb{R}^d$, $d$ is the number of spatial dimensions, and conservative state vector $Q_i$ and inviscid flux vectors $F$ are defined by

$$Q_i = \begin{pmatrix} \rho \\ \rho u_i \\ \rho v \\ \rho w \\ B_i \end{pmatrix}, \quad F_{ij} = \begin{pmatrix} \rho u_i u_j + (p + \frac{B_j^2}{2})\delta_{ij} - B_i B_j \\ u_j (\rho e + p + \frac{B_j^2}{2}) - (\mathbf{B} \cdot \mathbf{u}) B_j \\ u_j B_i - B_i u_j \end{pmatrix},$$

(2)

where the summation convention has been used and $\rho, p, e$ denote the density, pressure, and specific total energy of the fluid, respectively. $u_i$ and $B_i$ are the velocity of the flow and magnetic field in the coordinate direction $x_i$. This set of equations is completed by the equation of state

$$p = (\gamma - 1)\rho(e - \frac{1}{2}u_j u_j - \frac{B_j B_j}{2\rho}),$$

(3)

which is valid for perfect gas, $\rho e = \rho u^2/2 + p/(\gamma - 1) + B^2/2$, $\gamma$ is the ratio of the specific heats, and the constraint $\nabla \cdot \mathbf{B} = 0$. We chose to solve the MHD equations in a conservative form using a finite volume approach

$$\frac{\partial}{\partial t} \int_\Omega Q dV + \int_{\partial\Omega} \mathbf{F}(Q) \cdot \mathbf{n} dS = 0$$

(4)

where

$$\mathbf{F}(Q) \cdot \mathbf{n} = (\mathbf{u} \cdot \mathbf{n}) \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ B_x \\ B_y \\ B_z \\ H \end{pmatrix} + \tilde{P} \begin{pmatrix} 0 \\ n_x \\ n_y \\ n_z \\ 0 \\ 0 \\ 0 \end{pmatrix} - (\mathbf{B} \cdot \mathbf{n}) \begin{pmatrix} 0 \\ B_x \\ B_y \\ B_z \\ u \\ v \\ w \end{pmatrix}$$

(5)
\[ H = \rho h = \rho \left( e + \frac{P}{\rho} + \frac{B^2}{\rho} \right) \] (6)

\[ h = \frac{1}{2} u^2 + \gamma \frac{P}{(\gamma - 1)\rho} + \frac{B^2}{\rho} \] (7)

and \( \mathbf{n} = (n_x, n_y, n_z) \) represents the unit normal to the control volume face and \( S \) is the face area.

2.1. Solving the MHD Equations

2.1.1. Space Discretization

The equations are solved by a finite volume method utilizing the edge based scheme of [7]. For the control volume \( \Omega_I \), Eq.(4) can be written in a space-discretized form as:

\[ \frac{\partial \langle Q_I \rangle}{\partial t} = -\frac{1}{\Delta V_I} \sum_{\kappa=1}^{N} \Delta S_\kappa F_{\kappa} \left( \langle Q^R_\kappa \rangle, \langle Q^L_\kappa \rangle \right) \mathbf{n}_\kappa, \] (8)

an upwinded approximate flux from a linearized Riemann solver is used of the form

\[ \mathcal{F}(Q_f) \cdot \mathbf{n} = \frac{1}{2} \left[ \mathcal{F}(Q_L) + \mathcal{F}(Q_R) - \left| \tilde{A} \left( \tilde{Q} \right) \right| (Q_R - Q_L) \right] \cdot \mathbf{n} \] (9)

where \( \Delta S_\kappa \) is the \( \kappa \) face area of the control volume \( \Delta V_I \), \( N \) the number of faces enclosing the control volume, the Jacobian matrix \( \tilde{A} \equiv \frac{\partial \mathcal{F}}{\partial Q} \) using the appropriate average \( \tilde{Q} \) of the right \( Q_R \) and left \( Q_L \) states, and \( Q_f \) represents the conservative resolved state (the states at the faces). Eq.(8) states that the time rate of change of the volume averaged state \( \langle Q \rangle_I \) in the \( I^{th} \) volume is balanced by the sum of the area-averaged flux \( F_{\kappa} \) through the discrete boundary faces \( \kappa \). The subdomains \( \Omega_I \) may be represented by any arbitrarily shaped volume. For \textit{structured} methods, the control volume is typically a hexahedral with orthogonal Cartesian coordinates and the summation in Eq.(8) is over the six faces making up the hexahedral control volume. \textit{Structured} methods are based on a logical indexing of the data structure. Eqs(8) can also be solved using a general indexing system are usually referred to as \textit{unstructured} methods. The method employed in this paper uses a indexing scheme with a node averaged control volume. The global computational domain is subdivided into arbitrary shaped control volumes, or cells, that are enclosed by \( n \) faces \( \Delta S_k \).
2.1.2. Higher Spatial and Temporal Order

*Adapt3DMHD* uses a fully explicit Runge-Kutta time-stepping scheme. Defining

\[ R_I = \sum_{\kappa=1}^{N} \Delta S_{\kappa} \mathcal{F}_{I\kappa} \left( \langle Q \rangle_{I\kappa}^R, \langle Q \rangle_{I\kappa}^L \right) n_{\kappa} \]  \hspace{1cm} (10)

the residual accrued by summation of fluxes through the \( N \) faces of a volume surrounding a node \( I \), Eq.(8) becomes

\[ \frac{\partial \langle Q_I \rangle}{\partial t} = -\frac{1}{\Delta V_I} R_I \]  \hspace{1cm} (11)

Thus the timestepping strategy \[6\] follows

\[ Q_{I}^{(0)} = Q_{I}^{(n)} \]  \hspace{1cm} (12)

\[ Q_{I}^{(1)} = Q_{I}^{(0)} - \alpha_1 \frac{\Delta t_I}{\Delta V_I} R_{I}^{(0)} \]

\[ \ldots = \ldots \]

\[ Q_{I}^{(m-1)} = Q_{I}^{(0)} - \alpha_{m-1} \frac{\Delta t_I}{\Delta V_I} R_{I}^{(m-2)} \]

\[ Q_{I}^{(m)} = Q_{I}^{(0)} - \alpha_m \frac{\Delta t_I}{\Delta V_I} R_{I}^{(m-1)} \]

\[ Q_{I}^{(n+1)} = Q_{I}^{(m)} \]

where the superscript \( n \) denotes the time level, \( m \) the order of temporal accuracy, and the weighting factors are given by

\[ \alpha_k = \frac{1}{m - k + 1}, \quad k = 1, \ldots, m \]

The results presented here use \( m = 3 \) to insure 2\textsuperscript{nd} order in time \[8\].

The high order spatial reconstruction follows \[7\]. The MHD fluxes are obtained through a linearized Riemann solver using a Roe like flux differing scheme for the numerical flux

\[ \mathcal{F}_{ij} = \frac{1}{2} \left[ F_i^R + F_j^L \right] - \frac{1}{2} \left| A(Q_i^R, Q_j^L) \right| (Q_j^R - Q_i^L) , \]  \hspace{1cm} (13)
where $F^R_i = F(Q^R_i)$ and $F^L_j = F(Q^L_j)$. The upwinded biased reconstruction for $Q^R_i$ and $Q^L_j$ are

$$Q^R_i = Q_i + \frac{1}{4} [(1 - k)\Delta^L_i + (1 + k)(Q_j - Q_i)] ,$$  \hspace{1cm} (14)

$$Q^L_j = Q_j - \frac{1}{4} [(1 - k)\Delta^R_j + (1 + k)(Q_j - Q_i)] ,$$  \hspace{1cm} (15)

and the forward and back wards difference operators are just

$$\Delta^L_i = Q_i - Q_{i-1} = 2(\nabla Q)_i \cdot \ell^{ij} - (Q_j - Q_i) ,$$  \hspace{1cm} (16)

$$\Delta^R_j = Q_{j+1} - Q_j = 2(\nabla Q)_j \cdot \ell^{ij} - (Q_j - Q_i) ,$$  \hspace{1cm} (17)

where $\ell^{ij} = x_j - x_i$ is the length vector for edge $ji$. The parameter $k$ used here has a value $1/3$ and yields a third order-accurate upwind scheme, as least in linear 1D problems.

Since higher order schemes can lead to dispersive effects, particularly in the vicinity of steep gradients, we use a MUSCL [9] like limitor monotonicity scheme. Thus equations 14 and 15 are modified as

$$Q^R_i = Q_i + \frac{s_i}{4} [(1 - ks_i)\Delta^L_i + (1 + ks_i)(Q_j - Q_i)]$$  \hspace{1cm} (18)

$$Q^L_j = Q_j - \frac{s_j}{4} [(1 - ks_j)\Delta^R_j + (1 + ks_j)(Q_j - Q_i)] ,$$  \hspace{1cm} (19)

where $s$ is defined as

$$s_i = max \left[ 0, \frac{2\Delta^R_i(Q_j - Q_i) + \epsilon}{(\Delta^R_i)^2 + (Q_j - Q_i)^2 + \epsilon} \right]$$  \hspace{1cm} (21)

$$s_j = max \left[ 0, \frac{2\Delta^R_j(Q_j - Q_i) + \epsilon}{(\Delta^R_j)^2 + (Q_j - Q_i)^2 + \epsilon} \right]$$  \hspace{1cm} (22)

where $\epsilon$ is a small number to prevent division by zero in smooth regions of the flow.
There are serious errors that can be introduced into the solution of the MHD equations which do not occur in the solution of the Euler equations. An examination of the MHD momentum flux shows that the MHD momentum flux differs from the Euler momentum flux by the term \( \tilde{\mathbf{F}}_{ij} = \frac{B^2}{2} \delta_{ij} - B_i B_j \). This term is sometimes referred to as the MHD Maxwell stress tensor and its divergence is just the usual Lorentz force, \( -\mathbf{J} \times \mathbf{B} \). Since \( \mathbf{B} \cdot \mathbf{J} \times \mathbf{B} = 0 \) it follows that \( \mathbf{B} \cdot \nabla \cdot \tilde{\mathbf{F}} = 0 \) also. Since discretization errors can lead to \( \nabla \cdot \mathbf{B} \neq 0 \), it is not only necessary to insure \( \nabla \cdot \mathbf{B} = 0 \) numerically (the number one source of error; see 3.1) but it is also necessary to insure that \( \mathbf{B} \cdot \nabla \cdot \tilde{\mathbf{F}} = 0 \) by projecting out any component of \( \mathbf{B} \cdot \nabla \cdot \tilde{\mathbf{F}} \neq 0 \) parallel to \( \mathbf{B} \) (see 3.2). In addition, it is often the case that in regions where \( 2P/B^2 << 1 \) discretization errors can be large enough to cause negative pressures which is unacceptable in most applications of MHD. These errors are produced when computing the pressure from the total energy because the kinetic and
magnetic energies in some regions of the flow are very large and dominate the internal energy there. In section 3.3 we provide a short summary of our present approaches to solving these problems.

3.1. Solenoidal Condition

The approach we have chosen to insure \( \nabla \cdot \mathbf{B} = 0 \) is a technique known as the Hodge projection which yields a magnetic field that is divergence free to truncation. Hodge projection uses the fact that a vector \( \mathbf{B} \) can be decomposed into a solenoidal part and an irrotational part. In the case of the magnetic field vector we write the exact (analytical) magnetic field vector \( \mathbf{B}_e \) as the sum of two parts, the part obtained from the numerical solution for \( \mathbf{B}_i \) in equation (2), which we denote as \( \mathbf{B}_n \), and an irrotational correction \( \nabla \phi \), that is,

\[
\mathbf{B}_e = \mathbf{B}_n + \nabla \phi .
\] (23)

Since \( \nabla \cdot \mathbf{B}_e = 0 \) it follows that

\[
\nabla^2 \phi = -\nabla \cdot \mathbf{B}_n
\]

which can be solved using standard elliptical solver techniques subject to appropriate boundary conditions. Admittedly, our present algorithm for insuring the magnetic field is solenoidal is not the best but it works surprisingly well. In addition, the algorithm can only be used at every time level of a time step at considerable computational expense. Further, the technique, because it is global in nature can be dangerous from a physical point of view. For this reason we are investigating an approach used on structured meshes [2], which because of the duality of the MHD electric fields and their corresponding fluxes, permits the use of the upwinded Riemann fluxes to obtain solenoidal components of the magnetic field normal to the each face of the node control volume. These magnetic field normals can then be used to reconstruct \( \mathbf{B} \) on the control nodes that are solenoidal. However, even the staggered mesh approach does not eliminate the need for removing the field aligned frictional force or the negative pressure problem endemic in MHD codes that are used to study flows in low \( \beta \) plasmas.

We solve the spatial discretization of the Poisson equation using a finite element formulation, where a piecewise linear basis function is used to approximate the equation. The resulting system of the linear equations is solved using a conjugate gradient method. Both diagonal and LU-SGS preconditioners are implemented to accelerate the convergence of the Conjugate Gradient method.
3.2. Fictious Field Aligned Force

While it is the case that analytically
\[ \mathbf{B} \cdot \mathbf{J} \times \mathbf{B} = -\mathbf{B} \cdot \nabla \cdot \tilde{\mathbf{F}} = 0 \]
it is not the case numerically. It is not possible to simultaneously insure momentum conservation and to insure there are no fictious field aligned forces unless \( \nabla \cdot \mathbf{B} = 0 \) everywhere in the computational domain which is numerically difficult to insure. However, from our tests it can be shown that if \( \nabla \cdot \mathbf{B} = 0 \) to round-off, or to truncation, that the application of projecting out any component of \( \nabla \cdot \tilde{\mathbf{F}} \) parallel to \( \mathbf{B} \) leads to only small violations to total momentum conservation. The major numerical issue is how to split off the MHD momentum flux from the total momentum flux when using a high fidelity Reimann solver so that \( \nabla \cdot \tilde{\mathbf{F}} \) can be computed. To accomplish this we compute not only the resolved state fluxes but also the resolved state conserved quantities. Using the resolved state conserved quantities we construct the MHD momentum fluxes and subtract them from the total momentum fluxes. We then proceed as usual, that is, we update the conserved quantities but without the MHD momentum sources. We then use the MHD momentum fluxes to compute the change in momentum due to the Lorentz force
\[ \Delta \mathbf{M} = dt \frac{\Delta \tilde{\mathbf{F}}(q) \cdot \mathbf{n} dS}{\Delta V} \tag{24} \]
and correct \( \Delta \mathbf{M} \) so that \( \Delta \mathbf{M}_{\text{corrected}} = \Delta \mathbf{M} - (\mathbf{n}_b \cdot \Delta \mathbf{M}) \mathbf{n}_b \), where \( \mathbf{n}_b \) is a unit vector parallel to \( \mathbf{B} \). \( \Delta \mathbf{M}_{\text{corrected}} \) is then added back into the total momentum.

3.3. Positive Pressures

When the magnetic energy density, \( B^2/2 \), or the flow energy density, \( \rho u^2/2 \), is large compared to the internal energy \( p/(\gamma - 1) \) computing the pressure from \( p = (\gamma - 1)\rho(e - \frac{1}{2}\mathbf{u}^2 - \frac{\mathbf{B}^2}{2\rho}) \) can easily yield non-physical negative pressures. We employ a number of physical strategies to insure the pressure remains positive. Two of these are outlined in [1] both relying on the entropy density \( P/\rho^{\gamma-1} \) being a conserved quantity in regions of the flow that are incompressible. A third strategy in additional to transporting the MHD total energy \( \mathcal{E}_M = \mathcal{E}_e + \mathbf{B}^2/2 \) is to transport in addition the Euler energy \( \mathcal{E}_e = \rho u^2/2 + P/(\gamma - 1) \) and to correct the kinetic energy for the Lorentz force. To utilize this strategy it is first important to recognize that
the Lorentz force cannot directly affect the pressure so that the Lorentz force will only affect the kinetic energy in the Euler energy. To take advantage of this fact we use the well known fact that Euler energy equation is corrected for the Lorentz force by $\mathbf{u} \cdot \mathbf{J} \times \mathbf{B}$ so that

$$\frac{\partial \mathcal{E}_e}{\partial t} + \nabla \cdot [\mathbf{u}(\mathcal{E}_e + P)] = \mathbf{u} \cdot \mathbf{J} \times \mathbf{B} \tag{25}$$

and since

$$\mathbf{u} \cdot \mathbf{J} \times \mathbf{B} = \frac{1}{2} \rho \frac{d\mathbf{u}^2}{dt} \tag{26}$$

$\mathcal{E}_e$ can be corrected for the Lorentz force by first computing the updated kinetic energy without the effect of the Lorentz force, $\frac{1}{2} \rho \mathbf{u}_{WOLF}^2$, and then correcting the kinetic energy with the Lorentz force, $\frac{1}{2} \rho \mathbf{u}_{WLF}^2$, so that the corrected total Euler energy is corrected as

$$\mathcal{E}_e = \mathcal{E}_e + \frac{1}{2} \rho (\mathbf{u}_{WLF}^2 - \mathbf{u}_{WOLF}^2) \tag{27}$$

where $\mathbf{u}_{WLF}$ is the velocity corrected for the Lorentz force and $\mathbf{u}_{WOLF}$ is the velocity corrected without the Lorentz force. This approach permits the pressure to be computed without the need to substracting the magnetic energy density from the total energy.

### 3.4. Heat Conduction

In many space and solar phenomena thermal conduction plays a major role in the energy balance. To account for this we solve the electron heat conduction equation using implicit methods, which have the virtue of being unconditionally stable. The second order accurate Crank-Nicholson method is used for the time-accurate simulations, while the first order accurate implicit Euler scheme is used for the steady state problems. The spatial discretization is carried using a finite element formulation, where a piecewise linear basis function is used to approximate the heat conduction equation. The resulting system of the linear equations is solved using a conjugate gradient method. Both diagonal and LU-SGS preconditioners are implemented to accelerate the convergence of the Conjugate Gradient method.

$$n_e \frac{d(k_B T_e)}{dt} = -\nabla \cdot \mathbf{q}_e + Q_e \tag{28}$$
where
\[ q_e = -\kappa^e_\parallel \nabla (k_b T_e) - \kappa^e_\perp \nabla \perp (k_b T_e) - \kappa^e_\wedge n_b \times \nabla \perp (k_b T_e) \]  
(29)

\[ \kappa^e_\parallel = 3.2 \frac{n_e k_b T_e \tau_e}{m_e}, \ \kappa^e_\perp = 4.7 \frac{n_e k_b T_e}{m_e \Omega_{ce}^2 \tau_e}, \ \kappa^e_\wedge = 5 \frac{n_e k_b T_e}{m_e \Omega_{ce}}, \ \Omega = \frac{eB}{m_e c}, \ \tau_e = \frac{3 \sqrt{m_e (k_b T_e)^{3/2}}}{4 \sqrt{2 \pi n_e \lambda e^4}} \] (see [3] for additional details), \( Q_e \) represents the various heating and cooling functions depending on the physical problem at hand. Using the RHS of Eq.(28) we correct the energy density for the new electron temperature. We also carry \( T_e \) as an independent variable in order to use it to compute accurately source terms such as radiation losses etc. where the electron temperature is explicitly required.

![Figure 3: A cutting plane showing the temperature contours and magnetic lines resulting from a 3D simulation of the of the volume from 1-30R⊙ around the sun.](image-url)
4. Numerical Examples

In this section we provide results from a number of common, and not so common, MHD test problems. While our selection of problems is not exhaustive they are excellent examples that demonstrate the numerical approach we have chosen is fully capable of solving the MHD equations and providing results consistent and comparable to those solutions found with structured grid solvers.

4.1. Brio-Wu Shock Tube Problem

The Brio-Wu test Shock tube problem Brio and Wu [4] is the standard 1D test problem used to demonstrate the prowess of a MHD solver. It demonstrates whether the algorithm used to solve the ideal MHD equations is capable of properly characterizing various MHD structures. The initial left and right states are given by \( \rho_l = 1 \), \( u_l = v_l = 0 \), \( p_l = 1 \), \( B_{yl} = 1 \); and \( \rho_r = 0.125 \), \( u_r = v_r = 0 \), \( p_r = 0.1 \), \( B_{yr} = -1 \). Further, \( B_x = 0.75 \) and \( \gamma = 2 \). This problem tests wave properties of a particular MHD solver, because it involves a slow compound wave, two fast rarefaction waves, a contact discontinuity, and a slow shock wave. No AMR was used in this test nor was it necessary to use any divergence cleaning of the magnetic field. As a test problem it is rather limited because it is strictly a 1D problem.

Figure 4: The Brio-Wu Shock Tube (a) Density (b) Pressure (c) Vx-Component (d) Vy-Component and (e) By-component.
4.2. Tang-Orzag Problem

The Orszag-Tang MHD vortex problem (Orszag and Tang, 1979), is a straightforward 2D problem that uses periodic boundary conditions which has become a default test for MHD codes to examine the ability of a particular solver to deal with turbulence. In this problem a simple initial condition is imposed at time $t = 0$:

$$\mathbf{v} = v_0 (-\sin(2\pi y), \sin(2\pi x), 0), \mathbf{B} = B_0 (-\sin(2\pi y), \sin(4\pi x), 0), (x, y) \in [0, 1]^2,$$

where $B_0$ is chosen so that the ratio of the gas pressure to the RMS magnetic pressure is equal to $2\gamma$. We use an initial density, a speed of sound, and $V_0$ that are set to unity which in turn requires the initial pressure $p_0 = 1/\gamma$ and $B_0 = 1/\gamma$.

The development of vortex flow becomes increasingly complicated resulting from the nonlinear interactions of waves produced by the initial velocity field. A highly resolved simulation of this problem should produce two-dimensional MHD turbulence (see [5] for an example). Figures 5 shows density and magnetic field contours at $t = 0.5$. The non-linear flow pattern at this time shows that a number of strong non-linear waves have formed and passed through one another, causing turbulent eddies at all resolved spatial scales. This problem has also been used to compare spectral methods with finite element methods on an unstructured grid using AMR [5].

Figure 5: Density contours (left) and Magnetic pressure contours (right) in the Orszag-Tang MHD vortex problem at $t = 0.5$
4.3. Balsara-Spicer Rotor problem

The two-dimensional MHD rotor problem [2] was designed to study the onset and propagation of strong torsional Alfvén waves such as occurs in the sun. The computational domain is a unit square $[0, 1] \times [0, 1]$ with non-reflecting boundary conditions on all four sides. The initial conditions are given by

$$\rho(x, y) = \begin{cases} 
10 & r \leq r_0 \\
1 + 9f(r) & r_0 < r < r_1 \\
1 & r \geq r_1 
\end{cases}$$

$$u(x, y) = \begin{cases} 
-f(r)u_0(y - \frac{1}{2})/r_0 & r \leq r_0 \\
-f(r)u_0(y - \frac{1}{2})/r & r_0 < r < r_1 \\
0 & r \geq r_1 
\end{cases}$$

$$v(x, y) = \begin{cases} 
f(r)u_0(x - \frac{1}{2})/r_0 & r \leq r_0 \\
f(r)u_0(x - \frac{1}{2})/r & r_0 < r < r_1 \\
0 & r \geq r_1 
\end{cases}$$

$$p(x, y) = 1, \quad B_x(x, y) = \frac{5}{\sqrt{4\pi}}, \quad B_y(x, y) = 0,$$

where $r_0 = 0.1, r_1 = 0.115, \gamma = 1.4$. The initial set-up is a high density rotating disk at the center of the domain, surrounded by the ambient uniform density and pressure at rest. The rapidly spinning rotor not being in an equilibrium state due to the centrifugal forces spins with the given initial rotating velocity, the initially uniform magnetic field in $x$-direction will wind up the rotor. The rotor will be wrapped around by the magnetic field, and hence start launching torsional Alfvén waves into the ambient fluid. The angular momentum of the rotor will be shred by torsional Alfvén waves in later times. The circular rotor will be increasingly compressed by the build-up of the magnetic pressure around the rotor into an oval shape.
4.4. Solar Differential Rotation

This test problem is a more realistic test than the 2D rotor in section 4.3 because it is 3D and uses a realistic model of differential rotation. The initial magnetic field used was that of a 3D dipole centered within a sphere. The differential rotation flow pattern is imposed on the surface of the sphere as a boundary condition and the flow is allowed to wrap up the magnetic field. The angular momentum of the differentially rotating sphere launches torsional Alfvén waves which then propagate away from the sphere along the axis of symmetry of the rotating sphere which is the \( z \)-direction. As expected, the faster the rotation speed the tighter the wrapping of the magnetic fields became. However, unlike the rotor in section 4.3 the angular momentum transferred to the magnetic field in the form of torsional Alfvén waves can propagate out of the domain of the simulation. This minimizes any distortion of the sphere by the build up of magnetic pressure. In the case modeled here the sphere is a rigid body and cannot be distorted by magnetic pressure unlike the rotor in section 4.3. However, if we were to use ALE the shape of the sphere could respond to the Lorentz and centrifugal forces being generated.

This test problem used an inner sphere at \( R = 1 \) and an outer sphere at \( R = 30 \). The boundary conditions on the outer sphere were “free” while the inner sphere used a flow field given by

\[
egin{align*}
  u(x, y, z) &= -r \sin \theta \sin \phi \dot{\phi} \\
  v(x, y, z) &= r \sin \theta \cos \phi \dot{\phi} \\
  w(x, y, z) &= 0
\end{align*}
\]
where $\dot{\phi} = 2.886 \times 10^{-6} - 2.0 \times 10^{-6} \sin^2 \Theta - 0.370 \times 10^{-6} \sin^4 \Theta$ which is identical to that of the Sun except that a larger rotation speed was used, where $\Theta = \frac{\pi}{2} - \phi$ is the heliospheric latitude. The boundary condition on the magnetic field at the inner boundary required the normal component of $\mathbf{B}$ not to change while a zero derivative was imposed on $\rho$ and $P$.

Figure 7: Accelerated Differential Rotation of a Magnetized Sphere: left view in the x-z plane and on the right a view looking straight down the axis of rotation
4.5. 3D Magnetized Explosion Test Problem

This test is essentially a 3D exploding diaphragm. A spherical grid with an outer radius of 0.87 was used. The explosion is driven by a high pressure, $p = 1000$, region within a radius $r \leq 0.1$ expanding into a low pressure region of $p = 0.1$ with a constant density throughout both regions. A uniform magnetic field in the $z$-direction of magnitude 100 and a ratio of specific heats of $\gamma = 1.4$. The explosion results in an anisotropic expansion, the greatest expansion being at $z = 0$. Essentially the explosion forms a diamagnetic bubble. The velocity components $v_x$ and $v_y$ in the $x, y$ plane are anti-symmetric while the density and pressure are symmetric. The flow velocity $v_z$ in the $z$-direction is symmetric.

Figure 9: 3 Magnetized Explosion Test at $t = 0.003$ and xy plane (a) Density (b) Pressure (c) y-velocity (d) z-magnetic field.
Figure 10: 3D Magnetized Explosion Test at $t = 0.003$ and $zy$ plane (a) Density (b) Pressure (c) $y$-magnetic (d) $z$-magnetic field.

4.6. Magnetosphere

The magnetosphere of the earth represents a far superior test of an MHD code than any other 3D test problem that the authors are aware of because *in situ* empirical data exists which allows us to compare our MHD results to measured results. However, because we do not have a realistic ionospheric model coupled to the MHD code our simulation does not take account of the depletion of day-side magnetic flux by reconnection and its replenishing from the night-side reconnecting magneto-tail. Our test assumes a southward IMF with supersonic inflow of 400km/s with a density of $1.88 \times 10^{-23} \text{g/cm}^3$, pressure of $1.40 \times 10^{-12}$ and a magnetic field of $B_z = -0.5 \times 10^4$ gauss from the sun. The results presented here are the first of their kind because the grid used is a paraboloid of rotation as opposed to a rectangular box (see Figure( 11)).

A typical magnetospheric model must impose boundary conditions on seven surfaces, six sides of box, and the inner surface representing the boundary for the plasma sphere and/or the ionosphere. Because the paraboloid of rotation has only two surfaces plus the inner surface representing the boundary for the plasma sphere and/or the ionosphere implementation of boundary conditions is easy. The bullet shaped surface represents the supersonic inflow boundary while the flat boundary represents the supersonic outflow boundary. In addition, the inner boundary is a real sphere.

5. Concluding Remarks

We have described the development, validation, and application of an upwind finite element algorithm to the simulation of three dimensional compressible MHD flow.

A discontinuous Galerkin formulation based a Taylor basis has been presented for solving the compressible Euler equations. This formulation is
able to provide a unified framework, where the finite volume schemes can be recovered as special cases of the discontinuous Galerkin method by choosing reconstruction schemes to compute the derivatives, offer the insight why the DG methods are a better approach than the finite volume methods based on either TVD/MUSCL reconstruction or essentially non-oscillatory (ENO)/weighted essentially non-oscillatory (WENO) reconstruction, and has a number of distinct, desirable, and attractive features, which can be effectively used to address some of shortcomings of the DG methods. The developed method is used to compute a variety of both steady-state and time-accurate flow problems on arbitrary grids. The numerical results demonstrated the superior accuracy of this discontinuous Galerkin method in comparison with a second order finite volume method and a third order WENO method, indicating its promise and potential to become not just a competitive but simply a superior approach than its finite volume and ENO/WENO counterparts for solving flow problems of scientific and industrial interest.
The versatility of this DG method is also demonstrated in its ability to compute 1D, 2D, and 3D problems using the very same code, greatly alleviating the need and pain for code maintenance and upgrade.

References


Figure 13: Density in the z-x plane


Figure 14: Cutting Plane in the z-x plane


Figure 15: Magnetic field lines in the z-x plane

Figure 16: Field aligned current pattern resulting from a southward IMF: view from above north pole
Figure 17: Electric potential pattern and the resulting $\mathbf{E} \times \mathbf{B}$ drifts from a southward IMF: view from above north pole