<u>**Derivation of hemispherically integrated Joule heating**</u>

By Dibyendu Sur (Catholic University of America, CUA) from derivation by Lutz Rastaetter (CCMC, NASA GSFC) and assistance from Robert Robinson (CUA)

November 2024

This document shows that equating the hemispherically integrated Joule heating with the hemispheric integration of radial current density times the electric potential is valid provided that either the perpendicular electric current ($J_{\perp}$) or the electric potential (Φ) are zero along the low latitude boundary.

We know,

$$J_{\perp} = \Sigma_P \ . E \quad \text{and} \quad E = -\nabla\phi \tag{1}$$

The hemispherically integrated Joule heating (performing surface integral of local Joule heating values over one hemisphere) can be written as

$$\iint_{hem} \Sigma_P \ |E|^2 df = \iint_{hem} E(\Sigma_P \ .E\ )\, df = \iint_{hem} E.J_{\perp} df = \iint_{hem} (-\nabla\phi).J_{\perp}\, df$$

(using equation 1) (2)

As we know

$$\nabla.(\phi\, J_{\perp}) = J_{\perp}.\nabla\phi + \phi\,(\nabla.J_{\perp})$$

$$\text{Or} \qquad J_{\perp}.\nabla\phi = \phi\,(\nabla.J_{\perp}) - \nabla.(\phi\, J_{\perp}) \tag{3}$$

Taking surface integrals over one hemisphere on both sides of equation (3)

$$\iint_{hem} (-\nabla\phi).J_{\perp}\, df = -\iint_{hem} (\phi)(\nabla.J_{\perp})\, df + \iint_{hem} \nabla.(\phi\, J_{\perp}) df \tag{4}$$

As we know $J_R = -\nabla.J_{\perp}$ (Robinson et al., 2021), the first term of right-hand side of equation (4) becomes the following

$$\iint_{hem} (\phi)(-\nabla.J_{\perp})\, df = \iint_{hem} (\phi) J_R\, df = \iint_{hem} J_R\, \phi\, df$$

To evaluate the second term of equation (4) on the right side, we use the 2D Divergence theorem

$\iint_{hem} \nabla.(\phi J_{\perp})\, df = \int_C \ \phi\, J_{\perp} n\, ds$ (line integral at the boundary of the surface)

Where C is the closed curve or the boundary of the hemisphere

n is the unit vector normal to the plane of C

Thus, the hemispherically integrated Joule heating becomes

$$= \iint_{hem} J_R \phi \, df + \int_C \phi \, J_\perp n \, ds \qquad (5)$$

So, if at the low latitude boundary, either ϕ or $J_\perp$ becomes zero, the second term vanishes (which is true for all models used in this paper).

Thus, it can be shown,

$$\text{Hemispherically integrated Joule heatings} = \iint_{hem} J_R \phi \, df \qquad (6)$$

$\iint_{hem} J_R \phi \, df$ is available from CCMC based model simulations as one of the model outputs as the term JrPHI for the entire Earth. JrPHI_N or JrPHI_S are the integrated values for the northern or southern hemisphere, respectively.

(https://ccmc.gsfc.nasa.gov/VIS-DOCS/physical-variables/)

References:

Robinson, R. M., Zanetti, L., Anderson, B., Vines, S., & Gjerloev, J. (2021). Determination of auroral electrodynamic parameters from AMPERE field-aligned current measurements. *Space Weather, 19(4),* e2020SW002677. https://doi.org/10.1029/2020SW002677